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# Weak superadditivity of skew information

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**Abstract**
We revisit the superadditivity conjecture for skew information which has recently been disproved by Hansen (2007 *J. Stat. Phys.* **126** 643). We establish two weak forms of superadditivity which are conjectured to be optimal. Our results show that in a certain sense the superadditivity is true with '50% off'.

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## 1. Introduction

Let $\rho$ be a density operator (i.e., state) and $H$ an observable for some finite-dimensional quantum system. In quantum mechanics, it is important to construct various numerical quantities synthesizing physical information from the pair of $\rho$ and $H$. Prominent examples are the mean (expectation), variance and probabilities generated from the spectral decompositions.

From the informational perspective, a particularly interesting and significant notion in this context is the skew information

$$I(\rho, H) = -\frac{1}{2}\mathrm{tr}[\sqrt{\rho}, H]^2$$

introduced by Wigner and Yanase in their study of quantum measurement theory [16]. Here, the square bracket denotes a commutator, i.e., $[A, B] = AB - BA$. The skew information was originally introduced as a measure of information content and can be interpreted as a quantum extension of the classical Fisher information [2, 3, 8, 10, 11, 13]. The celebrated convexity of $I(\rho, H)$ in $\rho$, and, more generally, the Lieb convexity theorem for the Wigner–Yanase–Dyson information

$$I_\gamma(\rho, H) = -\frac{1}{2}\mathrm{tr}[\rho^\gamma, H][\rho^{1-\gamma}, H], \qquad \gamma \in (0, 1),$$

has played a remarkable role in quantum information theory [7, 12]. In particular, the first proof of strong subadditivity of quantum entropy and the monotonicity of quantum relative entropy are essentially based on this convexity [12].

Apart from the convexity, another important and intriguing issue for skew information is superadditivity concerning composite systems. Let $\rho$ be a bipartite density operator shared by parties $a$ and $b$, with respective marginals $\rho_a = \mathrm{tr}_b \rho$ and $\rho_b = \mathrm{tr}_a \rho$ (partial trace). Let $H_a$ and $H_b$ be observables for parties $a$ and $b$, respectively, then the superadditivity conjecture states that [16]

$$I(\rho, H_a \otimes \mathbf{1} + \mathbf{1} \otimes H_b) \geqslant I(\rho_a, H_a) + I(\rho_b, H_b). \tag{1}$$

Here $\mathbf{1}$ stands for the identity operator (depending on the system). This conjectured inequality seems very intuitive and is required by Wigner and Yanase [16], as well as by Lieb [7], as a necessary condition for the skew information to be a reasonable notion of information. Its classical analogy, the superadditivity of the classical Fisher information, was established by Carlen in 1991 [1]. However, the quantum world is not so simple, and the present status concerning the superadditivity of skew information may be summarized as follows:

(i) The conjecture is not true in its general form. Hansen constructed a numerical counterexample in 2007 [4]. Furthermore, Seiringer showed that the superadditivity, which holds for all *pure* bipartite states, cannot be extended to pure tripartite states [15].

(ii) The conjecture is true in many special cases, e.g., when $\rho$ is any pure state or certain mixed states [9]. Moreover, random numerical experiment shows that for two-qubit states the violations of superadditivity are relatively rare.

In this paper, we revisit the superadditivity issue and establish two weak forms of superadditivity property for the skew information in section 2. These results rescue the superadditivity in a certain sense and are conjectured to be optimal. In section 3, we construct a family of simple counterexamples to superadditivity and another conjectured inequality. Finally, section 4 is devoted to discussions.

## 2. Weak superadditivity

Since the original superadditivity conjecture (1) is not true, it is desirable to inquire as to what extent it can be modified. In this section, we prove two alternative forms of superadditivity.

First, note that if inequality (1) were true, then by replacing $H_b$ with $-H_b$ and noting that $I(\rho_b, -H_b) = I(\rho_b, H_b)$, we would have

$$I(\rho, H_a \otimes \mathbf{1} - \mathbf{1} \otimes H_b) \geqslant I(\rho_a, H_a) + I(\rho_b, H_b). \tag{2}$$

This inequality stands on an equal footing with inequality (1). By adding them together, we would obtain

$$I(\rho, H_a \otimes \mathbf{1} + \mathbf{1} \otimes H_b) + I(\rho, H_a \otimes \mathbf{1} - \mathbf{1} \otimes H_b) \geqslant 2(I(\rho_a, H_a) + I(\rho_b, H_b)). \tag{3}$$

We will show that although neither inequality (1) nor inequality (2) is always true (actually they are equivalent in the sense that if one is *always* true, then the other is also true), their combination, i.e., inequality (3), is always true. This implies that, for any fixed $\rho$, $H_a$ and $H_b$, at least one of the inequalities (1) and (2) has to be true.

**Proposition 1.** *It holds that*

$$I(\rho, H_a \otimes \mathbf{1} + \mathbf{1} \otimes H_b) + I(\rho, H_a \otimes \mathbf{1} - \mathbf{1} \otimes H_b) \geqslant 2(I(\rho_a, H_a) + I(\rho_b, H_b)).$$

**Proof.** By the definition, we have

$$\begin{aligned} I(\rho, H_a \otimes \mathbf{1} \pm \mathbf{1} \otimes H_b) &= -\tfrac{1}{2}\mathrm{tr}[\sqrt{\rho}, H_a \otimes \mathbf{1} \pm \mathbf{1} \otimes H_b]^2 \\ &= -\tfrac{1}{2}\mathrm{tr}([\sqrt{\rho}, H_a \otimes \mathbf{1}] \pm [\sqrt{\rho}, \mathbf{1} \otimes H_b])^2 \\ &= I(\rho, H_a \otimes \mathbf{1}) + I(\rho, \mathbf{1} \otimes H_b) \mp \mathrm{tr}[\sqrt{\rho}, H_a \otimes \mathbf{1}][\sqrt{\rho}, \mathbf{1} \otimes H_b]. \end{aligned}$$

Consequently, by adding the above identities, we have

$$I(\rho, H_a \otimes \mathbf{1} + \mathbf{1} \otimes H_b) + I(\rho, H_a \otimes \mathbf{1} - \mathbf{1} \otimes H_b) = 2(I(\rho, H_a \otimes \mathbf{1}) + I(\rho, \mathbf{1} \otimes H_b)).$$

But it is known that [7, 10]

$$I(\rho, H_a \otimes \mathbf{1}) \geqslant I(\rho_a, H_a), \qquad I(\rho, \mathbf{1} \otimes H_b) \geqslant I(\rho_b, H_b).$$

The desired inequality follows. □

Next, we prove the following alternative weak form of superadditivity.

**Proposition 2.** *It holds that*

$$I(\rho, H_a \otimes \mathbf{1} + \mathbf{1} \otimes H_b) \geqslant \tfrac{1}{2}(I(\rho_a, H_a) + I(\rho_b, H_b)). \tag{4}$$

We first prepare two lemmas.

**Lemma 1.** *Let $\rho$ be a bipartite density operator shared by parties a and b, with marginals $\rho_a = \mathrm{tr}_b\rho$ and $\rho_b = \mathrm{tr}_a\rho$. Let $\sigma$ be another such density operator with marginals $\sigma_a = \mathrm{tr}_b\sigma$ and $\sigma_b = \mathrm{tr}_a\sigma$. Let*

$$D(\rho, \sigma) = \mathrm{tr}(\sqrt{\rho} - \sqrt{\sigma})^2$$

*be the so-called (quantum) Hellinger distance as defined in [10]. Then*

$$D(\rho, \sigma) \geqslant D(\rho_a, \sigma_a), \qquad D(\rho, \sigma) \geqslant D(\rho_b, \sigma_b).$$

*This is actually a particular instance of monotonicity of the general monotonic metrics [13], and also follows readily from theorem 2 in [10] if we note that $D(\rho, \sigma) = 2 - 2A(\rho, \sigma)$ with $A(\rho, \sigma) = \mathrm{tr}\sqrt{\rho}\sqrt{\sigma}$.*

**Lemma 2.** *Let $\rho$ be a density operator and $H$ an observable. Let $\rho(t) = \mathrm{e}^{-\mathrm{i}tH}\rho\,\mathrm{e}^{\mathrm{i}tH}$, $t \in R$, then for small $|t|$, we have the following asymptotic expansion:*

$$D(\rho(t), \rho) = \tfrac{1}{2}I(\rho, H)t^2 + o(t^2).$$

**Proof.** Apparently, since $\rho(0) = \rho$, we have $D(\rho(t), \rho)|_{t=0} = 0$. Next, from $\rho(t) = \mathrm{e}^{-\mathrm{i}tH}\rho\,\mathrm{e}^{\mathrm{i}tH}$, we have $\sqrt{\rho(t)} = \mathrm{e}^{-\mathrm{i}tH}\sqrt{\rho}\,\mathrm{e}^{\mathrm{i}tH}$, and thus

$$\frac{\partial}{\partial t}\sqrt{\rho(t)} = \mathrm{i}\,\mathrm{e}^{-\mathrm{i}tH}[\sqrt{\rho}, H]\,\mathrm{e}^{\mathrm{i}tH}.$$

Consequently,

$$\begin{aligned}\frac{\partial}{\partial t}D(\rho(t), \rho) &= -2\frac{\partial}{\partial t}\mathrm{tr}\sqrt{\rho(t)}\sqrt{\rho} \\ &= -2\mathrm{i}\cdot\mathrm{tr}\,\mathrm{e}^{-\mathrm{i}tH}[\sqrt{\rho}, H]\,\mathrm{e}^{\mathrm{i}tH}\sqrt{\rho}.\end{aligned}$$

Putting $t = 0$, we obtain

$$\frac{\partial}{\partial t}D(\rho(t), \rho)|_{t=0} = -2\mathrm{i}\cdot\mathrm{tr}[\sqrt{\rho}, H]\sqrt{\rho} = 0.$$

Moreover, by theorem 3(i) in [10], we have

$$\frac{\partial^2}{\partial t^2}D(\rho(t), \rho)|_{t=0} = \frac{1}{2}I(\rho, H).$$

The desired expansion follows.

Now, we proceed to the proof of proposition 2. Let $H = H_a \otimes \mathbf{1} + \mathbf{1} \otimes H_b$, and for $t \in R$, put

$$\rho(t) = \mathrm{e}^{-\mathrm{i}tH} \rho\, \mathrm{e}^{\mathrm{i}tH}, \qquad \rho_a(t) = \mathrm{e}^{-\mathrm{i}tH_a} \rho_a\, \mathrm{e}^{\mathrm{i}tH_a},$$

then

$$\rho_a(t) = \mathrm{tr}_b \rho(t).$$

From lemma 1, we have

$$D(\rho(t), \rho) \geqslant D(\rho_a(t), \rho_a).$$

On the other hand, from lemma 2, for small $|t|$, we have the asymptotic expansions

$$D(\rho(t), \rho) = \tfrac{1}{2} I(\rho, H)t^2 + o(t^2) \qquad D(\rho_a(t), \rho_a) = \tfrac{1}{2} I(\rho_a, H_a)t^2 + o(t^2).$$

Consequently, we must have

$$I(\rho, H) \geqslant I(\rho_a, H_a).$$

Similarly, we have

$$I(\rho, H) \geqslant I(\rho_b, H_b).$$

Now inequality (4) follows from combining the above two inequalities. □

**Remark 1.** Proposition 2 can also be proved by the use of the advanced theory of monotonic metrics, which is developed by Petz [13]. Our method is elementary and direct. Furthermore, proposition 2 can be generalized to any monotonic quantum Fisher information.

**Remark 2.** From the proof of proposition 2, we know that

$$I(\rho, H_a \otimes \mathbf{1} + \mathbf{1} \otimes H_b) \geqslant I(\rho_a, H_a).$$

On the other hand, we have

$$I(\rho, H_a \otimes \mathbf{1}) \geqslant I(\rho_a, H_a).$$

One naturally wants to know whether

$$I(\rho, H_a \otimes \mathbf{1} + \mathbf{1} \otimes H_b) \geqslant I(\rho, H_a \otimes \mathbf{1}). \tag{5}$$

We will demonstrate that this last inequality is not true in general in the next section.

We believe that proposition 2 is optimal in the following sense:

**Conjecture 1.** *Let $\beta$ be the largest positive constant such that*

$$I(\rho, H_a \otimes \mathbf{1} + \mathbf{1} \otimes H_b) \geqslant \beta(I(\rho_a, H_a) + I(\rho_b, H_b))$$

*for any density operator $\rho$ and observables $H_a$ and $H_b$, that is,*

$$\beta = \inf \frac{I(\rho, H_a \otimes \mathbf{1} + \mathbf{1} \otimes H_b)}{I(\rho_a, H_a) + I(\rho_b, H_b)}$$

*where the inf is over all $\rho$, $H_a$ and $H_b$ in finite dimensions and $\frac{0}{0}$ is understood to be 1. Then $\beta = \frac{1}{2}$.*

From Hansen's counterexample [4], as well as the simple analytical counterexamples in the next section, we know that $\beta < 1$. On the other hand, from proposition 2, we know that $\beta \geqslant \frac{1}{2}$. We have also performed extensive numerical studies in the two-qubit cases. The following example shows that at least $\beta \leqslant 0.6749$.

**Example 1.** Consider the two-qubit density operator

$$\rho = \frac{1}{270}\begin{pmatrix} 80 & 8 & 30 & 80 \\ 8 & 30 & 1 & 8 \\ 30 & 1 & 80 & 30 \\ 80 & 8 & 30 & 80 \end{pmatrix} = \begin{pmatrix} 0.2963 & 0.0296 & 0.1111 & 0.2963 \\ 0.0296 & 0.1111 & 0.0037 & 0.0296 \\ 0.1111 & 0.0037 & 0.2963 & 0.1111 \\ 0.2963 & 0.0296 & 0.1111 & 0.2963 \end{pmatrix}$$

acting on Hilbert space $C^2 \otimes C^2$, and observables

$$H_a = \begin{pmatrix} 13 & 1 \\ 1 & 0 \end{pmatrix}, \qquad H_b = \begin{pmatrix} 0 & 1 \\ 1 & 13 \end{pmatrix}.$$

The eigenvalues of $\rho$ are

$$\lambda_1 = 0, \quad \lambda_2 = 0.1068, \quad \lambda_3 = 0.2299, \quad \lambda_4 = 0.6633$$

with corresponding eigenvectors

$$|\psi_1\rangle = \begin{pmatrix} 0.7071 \\ 0 \\ 0 \\ -0.7071 \end{pmatrix}, \qquad |\psi_2\rangle = \begin{pmatrix} 0.0766 \\ -0.9916 \\ -0.0704 \\ 0.0766 \end{pmatrix},$$

$$|\psi_3\rangle = \begin{pmatrix} -0.2721 \\ -0.1072 \\ 0.9168 \\ -0.2721 \end{pmatrix}, \qquad |\psi_4\rangle = \begin{pmatrix} 0.6481 \\ 0.0722 \\ 0.3932 \\ 0.6481 \end{pmatrix}.$$

The square root of $\rho$ is given by

$$\sqrt{\rho} = \begin{pmatrix} 0.3796 & 0.0273 & 0.0862 & 0.3796 \\ 0.0273 & 0.3311 & -0.0012 & 0.0273 \\ 0.0862 & -0.0012 & 0.5305 & 0.0862 \\ 0.3796 & 0.0273 & 0.0862 & 0.3796 \end{pmatrix}.$$

Note that

$$H = H_a \otimes \mathbf{1} + \mathbf{1} \otimes H_b = \begin{pmatrix} 13 & 1 & 1 & 0 \\ 1 & 26 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 13 \end{pmatrix}.$$

It can be evaluated that

$$I(\rho, H) = -\tfrac{1}{2}\mathrm{tr}[\sqrt{\rho}, H]^2 = 2.8224.$$

On the other hand, the marginal states (partial traces) are given by

$$\rho_a = \begin{pmatrix} 0.4074 & 0.1407 \\ 0.1407 & 0.5926 \end{pmatrix}, \qquad \rho_b = \begin{pmatrix} 0.5926 & 0.1407 \\ 0.1407 & 0.4074 \end{pmatrix},$$

and it can be evaluated that

$$I(\rho_a, H_a) = I(\rho_b, H_b) = 2.0909.$$

Therefore, we conclude that

$$I(\rho, H) \approx 0.6749 \times (I(\rho_a, H_a) + I(\rho_b, H_b))$$

and thus $\beta \leqslant 0.6749$. We conjecture that $\beta = \frac{1}{2}$.

## 3. Simple counterexamples

In this section, we will construct a family of analytical counterexamples to the superadditivity inequality (1). These counterexamples are somewhat simpler than the remarkable original example of Hansen [4]. We will also construct a counterexample to the conjectured inequality (5).

**Example 2.** Consider a two-qubit system with Hilbert space $C^2 \otimes C^2$. Let $n > 2$ and take

$$\rho = \frac{1}{n}\begin{pmatrix} n-2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad H_a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad H_b = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We will show that

$$I(\rho, H_a \otimes \mathbf{1} + \mathbf{1} \otimes H_b) < I(\rho_a, H_a) + I(\rho_b, H_b)$$

for large $n$.

First, we evaluate $I(\rho_a, H_a)$ and $I(\rho_b, H_b)$. Clearly,

$$\rho_a = \frac{1}{n}\begin{pmatrix} n-1 & 0 \\ 0 & 1 \end{pmatrix}$$

which is already diagonal. Thus,

$$\sqrt{\rho_a} = \frac{1}{\sqrt{n}}\begin{pmatrix} \sqrt{n-1} & 0 \\ 0 & 1 \end{pmatrix},$$

and

$$[\sqrt{\rho_a}, H_a] = \frac{1}{\sqrt{n}}\begin{pmatrix} 0 & \sqrt{n-1}-1 \\ -\sqrt{n-1}+1 & 0 \end{pmatrix},$$

form which we readily obtain

$$I(\rho_a, H_a) = -\frac{1}{2}\mathrm{tr}[\sqrt{\rho_a}, H_a]^2 = \frac{1}{n}(n - 2\sqrt{n-1}). \tag{6}$$

Similarly, we have

$$I(\rho_b, H_b) = \frac{1}{n}(n - 2\sqrt{n-1}). \tag{7}$$

Next, we evaluate $I(\rho, H)$ with $H = H_a \otimes \mathbf{1} + \mathbf{1} \otimes H_b$. Since the spectral decomposition of $\rho$ can be written as

$$\rho = \sum_j \lambda_j |\psi_j\rangle\langle\psi_j|$$

with eigenvalues $\lambda_1 = \frac{n-2}{n}$, $\lambda_2 = \frac{2}{n}$, $\lambda_3 = \lambda_4 = 0$ and eigenvectors

$$|\psi_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \qquad |\psi_2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \qquad |\psi_3\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \qquad |\psi_4\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

we readily obtain

$$\sqrt{\rho} = \sum_j \sqrt{\lambda_j}|\psi_j\rangle\langle\psi_j| = \frac{1}{\sqrt{2n}}\begin{pmatrix} \sqrt{2(n-2)} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Note that

$$H = H_a \otimes \mathbf{1} + \mathbf{1} \otimes H_b = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix},$$

we have

$$[\sqrt{\rho}, H] = \frac{1}{\sqrt{2n}} \begin{pmatrix} 0 & N & N & 0 \\ -N & 0 & 0 & 2 \\ -N & 0 & 0 & 2 \\ 0 & -2 & -2 & 0 \end{pmatrix}$$

where $N = \sqrt{2(n-2)} - 2$. Consequently,

$$I(\rho, H) = -\frac{1}{2}\mathrm{tr}[\sqrt{\rho}, H]^2 = \frac{1}{n}(2n - 4\sqrt{2(n-2)} + 4).$$

From equations (6) and (7), we have

$$I(\rho_a, H_a) + I(\rho_b, H_b) = \frac{1}{n}(2n - 4\sqrt{n-1}).$$

Therefore, when $n > 10$, we have

$$I(\rho, H) < I(\rho_a, H_a) + I(\rho_b, H_b),$$

which contradicts the superadditivity inequality (1).

**Example 3.** The bipartite density operator $\rho$ and the observable $H_a$ are the same as in example 2, and the observable $H_b$ is modified to

$$H_b = \begin{pmatrix} 0 & x \\ x & 0 \end{pmatrix}, \qquad x \in R.$$

Now

$$H = H_a \otimes \mathbf{1} + \mathbf{1} \otimes H_b = \begin{pmatrix} 0 & x & 1 & 0 \\ x & 0 & 0 & 1 \\ 1 & 0 & 0 & x \\ 0 & 1 & x & 0 \end{pmatrix}.$$

Direct calculations yield

$$[\sqrt{\rho}, H] = \frac{1}{\sqrt{2n}} \begin{pmatrix} 0 & X & Y & 0 \\ -X & 0 & 0 & x+1 \\ -Y & 0 & 0 & x+1 \\ 0 & -x-1 & -x-1 & 0 \end{pmatrix}$$

where

$$X = \sqrt{2(n-2)}x - x - 1, \qquad Y = \sqrt{2(n-2)} - x - 1.$$

Consequently,

$$\begin{aligned} I(\rho, H) &= -\frac{1}{2}\mathrm{tr}[\sqrt{\rho}, H]^2 \\ &= \frac{1}{2n}(X^2 + Y^2 + 2(x+1)^2) \\ &= \frac{1}{n}((n - \sqrt{2(n-2)})x^2 + (4 - 2\sqrt{2(n-2)})x + n - \sqrt{2(n-2)}). \end{aligned} \tag{8}$$

First, put $x = 0$, we obtain

$$I(\rho, H_a \otimes \mathbf{1}) = \frac{1}{n}(n - \sqrt{2(n-2)}). \tag{9}$$

Next, we take

$$x = \frac{-2 + \sqrt{2(n-2)}}{n - \sqrt{2(n-2)}},$$

then

$$I(\rho, H) = \frac{1}{n}\left(n - \sqrt{2(n-2)} - \frac{(2 - \sqrt{2(n-2)})^2}{n - \sqrt{2(n-2)}}\right). \tag{10}$$

Comparing equations (9) and (10), we see that

$$I(\rho, H) < I(\rho, H_a \otimes \mathbf{1})$$

which serves as a counterexample to inequality (5). On the other hand, the reverse of inequality (5) cannot be true neither as can be seen by taking $x = 1$ in equation (8), then

$$I(\rho, H) = \frac{1}{n}(2n - 4\sqrt{2(n-2)} + 4)$$

which is larger than $I(\rho, H_a \otimes \mathbf{1})$.

## 4. Discussions

Before Hansen's counterexample, the superadditivity conjecture was largely believed to be true, and many particular cases were established. The intuition advertised by Wigner and Yanase themselves for such a conjecture is that when a composite system is separated into two parties, the correlations between them are lost, and thus the skew information decreases [16]. In particular, this is true in both the classical cases and for pure states in the quantum cases. It is further proved for many mixed states [9]. However, the quantum world is often counterintuitive and subtle, and our present results illustrate, in some sense, that the superadditivity is true with '50% off'.

The skew information is just one variant of quantum extensions of the classical Fisher information, which in turn is essentially the unique infinitesimal metric form in the space of probabilities. There are a variety of other variants of quantum Fisher information which are important in quantum estimation theory [2, 5, 6, 13, 14]; it would be worth investigating their superadditivity and implications for quantum information theory.

We do not know if there is any analytical tool to investigate the conjecture that $\beta = \frac{1}{2}$. It seems that it can only be resolved by numerical studies. We also conjecture that $\beta = \frac{1}{2}$ is not achievable in finite dimensions.

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